

Ground-State Entanglement Bound for Quantum Energy Teleportation of General Spin-Chain Models

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Abstract

Many-body quantum systems in the ground states have zero-point energy due to the uncertainty relation. In many cases, the system in the ground state accompanies spatially-entangled energy density fluctuation via the noncommutativity of the energy density operators, though the total energy takes a fixed value, i.e. the lowest eigenvalue of the Hamiltonian. Quantum energy teleportation (QET) is protocols for extraction of the zero-point energy out of one subsystem using information of a remote measurement of another subsystem. From an operational viewpoint of protocol users, QET can be regarded as an effective rapid energy transportation without breaking all physical laws including causality and local energy conservation. In the protocols, the ground-state entanglement plays a crucial role. In this paper, we show analytically for a general class of spin-chain systems that the entanglement entropy is lower bounded by a positive quadratic function of the teleported energy between the regions of a QET protocol. This supports a general conjecture that ground-state entanglement is an evident physical resource for energy transportation in the context of QET. The result may also deepen our understanding of the energy density fluctuation in condensed matter systems from a new perspective of quantum information theory.

1 Introduction

Many-body quantum systems including quantum fields have zero-point energy of quantum fluctuation in ground states due to the uncertainty relations. According to the passivity argument of the ground state [1], an arbitrary non-trivial local operation on the ground state does not cause the extraction of this energy but leads to the injection of additional energy into the system by exciting the zero-point fluctuation. This is because the local operation inevitably yields a different state of the system from the ground state as the lowest energy state and, thereby, the post-operation state possesses excitation energy. Therefore, in a fixed region, zero-point energy is useless for a single experimenter. In quantum field theory, expectation value of energy density operator in the ground state (vacuum state) is usually renormalized to zero by subtracting a divergent term corresponding to the zero-point fluctuation. This expresses that the system in the ground state represents *nothing* in physics and does not have any useful energy. However, the zero-point energy of many-body systems indeed becomes available and can be extracted if two separate experimenters (for example, Alice and Bob) are able to perform local operations and classical communication (LOCC) for a quantum system that possesses an entangled ground state [2] [3]. One of the key points of this energy extraction is a fact that many-body systems in the ground states often accompany spatially-entangled energy density fluctuation via the noncommutativity of the energy density operators, though the total energy takes a fixed value, i.e. the lowest eigenvalue of the Hamiltonian. Thus it is possible to perform in a spatial region an indirect measurement of the energy density fluctuation in a separate region by use of the ground-state entanglement. First, Alice performs a local measurement of quantum fluctuation of one subsystem in the ground state. Because of passivity, her measurement device excites zero-point fluctuation in her region and injects energy to the system. At the expense of measurement energy consumption, she obtains information about the quantum fluctuation and then announces it to Bob, who is in a distant region, with a light velocity that is much faster than the excitation propagation velocity of the system. It is of significance to note that the measurement result includes some information about the zero-point fluctuation of the subsystem in Bob's region via the ground-state entanglement. Based on the information, Bob can devise a strategy to suppress

the zero-point fluctuation. This enables him to extract the excess energy out of the subsystem in a local ground state much before the excitation resulting from the energy injected by Alice reaches Bob's region. Simultaneously, the suppression of zero-point fluctuation locally generates a region with a smaller value of energy density than that of the ground state in the system, which compensates for the energy extraction by his operation, in accordance with the local energy conservation law. From an operational viewpoint of the protocol users, the energy injected to the system in the ground state by Alice can be regarded as input of the protocol and the energy extracted from the subsystem in the local ground state by Bob as output. Hence, it is, in effect, energy transportation from Alice to Bob, though it is a one-time transfer for each entangled state just like the quantum information transfer by conventional protocols of quantum teleportation [4]. Thus, this new protocol is referred to as quantum energy teleportation (QET). In contrast to the case of QET protocols, by using the standard protocols of quantum teleportation [4], it is impossible to extract and utilize zero-point energy in its receiver region. In an arbitrary QET protocol, the amount of energy extracted by Bob is less than that injected by Alice, and it becomes smaller as the distance between them increases. QET has not been experimentally verified yet, but a realistic experiment, which may be achievable with present technology, has been proposed that uses quantum Hall edge currents [5]. QET affords not only future development of quantum technology but also various applications for fundamental physics. For example, this sheds a new light on entanglement in condensed matter systems from a viewpoint of local energy density fluctuation. Besides, QET may become a new available tool of the quantum Maxwell's demons who observe local quantum fluctuations of an interacting many-body system at the zero temperature and lead us to an extended paradigm of quantum information thermodynamics. In the past works about the demons [6], interactions between subsystems that the demon watches is assumed to be negligibly small. Thus the ground-state entanglement has not been taken account of even in the low temperature case. However, QET enables the demon to perform indirect measurements using the ground-state entanglement in order to extract work as a new tool. QET also has a close relation to a local-cooling problem of quantum many-body systems. A local measurement of zero-point fluctuation in a subsystem is generally accompanied by energy injection to the system and yields an excited state. Then a natural question arises. Soon after the energy injection, can we retrieve all the injected energy using only local operations on the measured subsystem?

The answer is no, and some residual energy is unavoidable in the system for any local-cooling procedure [2]. The reason for the residual energy is that the local measurement breaks a part of the ground-state entanglement and the broken entanglement cannot be restored by local operations. It turns out that the residual energy is lower bounded by the total amount of teleported energy via QET by use of the information of the local measurement [2]. Moreover, QET has been recently applied to black hole physics and gives a new method for decreasing area of the event horizon [7], just like the Hawking radiation [8]. Let us imagine that a measurement of quantum fields outside a massive black hole provides information about the quantum fluctuation. Because the pre-measurement state of the quantum fields can be approximated by the usual Minkowski vacuum state in the flat spacetime, positive-energy wave packets of the fields are generated during the measurement due to the passivity argument. Assume that the black hole absorbs the wave packets. Then, very significantly, we are capable of retrieving a part of the absorbed energy outside the horizon by QET. Using the measurement information, negative energy wave packets can be generated outside the horizon by extracting positive energy out of the zero-point fluctuation of the fields. The negative energy of the wave packets propagates across the event horizon of the black hole and may pair-annihilate with positive energy of matter previously falling inside the black hole. Hence this QET process is a phenomenon similar to the spontaneous emission of Hawking radiation which is often referred to as the energy tunneling out of black holes [8] [9]. The energy retrieval yields a decrease in the horizon area, which is proportional to the entropy of the black hole. This result may provide a profound suggestion about the origin of black hole entropy from a viewpoint of information theory. QET is one of promising tools in physics and will increase its advantage in various fields of research.

If we do not have any distant-region information via ground-state entanglement, no energy can be teleported by the QET protocols. It seems very plausible that the amount of teleported energy is closely related to the amount of the ground-state entanglement. Thus, the conjecture is possible that QET with a large amount of teleported energy generally requires a large amount of the entanglement as an evident physical resource. Interestingly this conjecture has been partially verified for two specific models. For a two-qubit model [10], it has been analytically shown that the amount of ground-state entanglement breaking by a local measurement of one qubit is lower bounded by a positive value that is proportional to the maximum

amount of energy teleported from the measured qubit to another qubit. For a harmonic chain model [11], a similar relation between entanglement consumption of local measurement and amount of teleported energy has been found by numerical analysis. In this paper, we show analytically, for a general class of spin-chain models, that the ground-state entanglement entropy is lower bounded by a positive quadratic function of the teleported energy between the regions of a QET protocol. This general inequality strongly supports the conjecture mentioned above. The result may also deepen our understanding of the energy density fluctuation in condensed matter systems from a new perspective of quantum information theory. In section 2, a brief review of QET is provided. In section 3, the entanglement bounds in the context of QET are given and analyzed. In the last section, summary and discussion are provided.

2 Formula for Energy Teleported by QET

In this section, a brief review of QET is provided. Let us consider a general model of a spin chain with nearest neighbor interaction. Assume that the ground state $|g\rangle$ is a pure non-degenerate state. The model is nonrelativistic, and the excitation propagation velocity of the system is assumed to be much smaller than the velocity of light. The dimension of the sub-Hilbert space for each spin is assumed to be finite. The energy density operator at site n is a Hermitian operator and takes the following form:

$$T_n = X_n - \frac{1}{2} \sum_l \left(g_{n-1/2}^{(l)} Y_{n-1}^{(l)} Y_n^{(l)} + g_{n+1/2}^{(l)} Y_n^{(l)} Y_{n+1}^{(l)} \right),$$

where X_n and $Y_n^{(l)}$ are local operators acting on a sub-Hilbert space at spin site n and $g_{n+1/2}^{(l)}$ are coupling constants. The total Hamiltonian of the system is given by the total sum of energy density operators:

$$H = \sum_n T_n = \sum_n X_n - \sum_{n,l} g_{n+1/2}^{(l)} Y_n^{(l)} Y_{n+1}^{(l)}.$$

Because we later focus on the difference between pre-operation energy and post-operation energy, we are able to assume, without changing the physics of the system, that the expectation value of T_n for the ground state $|g\rangle$ is

zero as a useful reference point; that is,

$$\langle g|T_n|g\rangle = 0, \quad (1)$$

when an appropriate constant is subtracted from each X_n . Because the energy eigenvalue of the ground state E_g is computed as

$$E_g = \langle g|H|g\rangle = \sum_n \langle g|T_n|g\rangle,$$

Eq. (1) also implies that E_g is set to zero by subtracting a constant from the original Hamiltonian:

$$H|g\rangle = 0. \quad (2)$$

Thus the Hamiltonian is a non-negative operator:

$$H \geq 0.$$

It is worthwhile here to stress that, when T_n do not commute with each other, T_n can take negative eigenvalues and, thereby, negative average values even though the total sum of T_n , namely the Hamiltonian H , is non-negative. For example, the energy density operator at site n of the Ising model with transverse external magnetic field b can be naturally introduced as

$$T_n = X_n - \frac{g}{2}Y_{n-1}Y_n - \frac{g}{2}Y_nY_{n+1},$$

where g is a real Ising coupling constant and the local operators are defined as

$$\begin{aligned} X_n &= b\sigma_n^z - \varepsilon_n, \\ Y_n &= \sigma_n^x, \end{aligned}$$

with irrelevant real constants ε_n . The operator σ_n^x (σ_n^z) is the x (z)-component of Pauli operator at site n . By using the substitution $\varepsilon_g = \sum_n \varepsilon_n$, the total Hamiltonian takes the standard form

$$\sum_n T_n = b \sum_n \sigma_n^z - g \sum_n \sigma_n^x \sigma_{n+1}^x - \varepsilon_g.$$

When ε_n is selected properly, Eq. (1) and Eq. (2) hold. In spite of the non-negativeness of H , T_n has negative average values except the cases with very specific values of the ratio g/b [2].

In the case of the QET protocol, Alice stays at site $n = n_A$ and Bob stays at $n = n_B$. Let us assume here that Alice is separated enough from Bob and the site distance between them satisfies

$$|n_A - n_B| \geq 3. \quad (3)$$

This condition guarantees local property of their operations in the QET protocols. In the first step, Alice locally performs a general measurement (POVM measurement) [17]. The measurement operator is given by $M_A(\mu)$, which is a local operator at site n_A and satisfies the normality condition

$$\sum_{\mu} M_A^{\dagger}(\mu) M_A(\mu) = 1. \quad (4)$$

The POVM of this measurement is written as

$$\Pi_A(\mu) = M_A^{\dagger}(\mu) M_A(\mu). \quad (5)$$

After the measurement yielding a result μ , its corresponding post-measurement state is given by

$$|\Psi_1(\mu)\rangle = \frac{1}{\sqrt{p_A(\mu)}} M_A(\mu) |g\rangle,$$

where $p_A(\mu)$ is the emergent probability of μ and is calculated as

$$p_A(\mu) = \langle g | \Pi_A(\mu) | g \rangle. \quad (6)$$

The average post-measurement state is provided by

$$\rho_1 = \sum_{\mu} p_A(\mu) |\Psi_1(\mu)\rangle \langle \Psi_1(\mu)|. \quad (7)$$

During the measurement, positive energy (E_A) is undoubtedly injected into the system because of passivity [1], and it is evaluated as

$$E_A = \text{Tr} [H \rho_1] - \langle g | H | g \rangle.$$

In the second step of the protocol, Alice announces the measurement result μ to Bob via a classical channel. Because the model is nonrelativistic, the time duration of communication and time evolution of the system can be omitted by assuming that the communication speed is the velocity of light. Thus, the information is received by Bob much before the excitation resulting

from the energy injected by Alice reaches Bob's region. In the third step, Bob performs a μ -dependent local operation on a spin at $n = n_B$ in a local ground state with zero average energy. The unitary operator is given by

$$U_B(\mu) = \exp(-i\theta(\mu) G_B(\mu)), \quad (8)$$

where $G_B(\mu)$ is a generally μ -dependent Hermitian local operator at $n = n_B$ and $\theta(\mu)$ is a real constant that is dependent on μ and is usually fixed so as to maximize Bob's energy gain from QET. After the operation, the state corresponding to the result μ is given by

$$|\Psi_2(\mu)\rangle = \frac{1}{\sqrt{p_A(\mu)}} U_B(\mu) M_A(\mu) |g\rangle \quad (9)$$

and the average post-operation state is given by

$$\rho_2 = \sum_{\mu} p_A(\mu) |\Psi_2(\mu)\rangle \langle \Psi_2(\mu)|. \quad (10)$$

In a specific setting of $\theta(\mu)$ and $G_B(\mu)$ for QET, the total energy of the system decreases during the operation. The local energy conservation law ensures that this loss in energy of the system is equal to Bob's energy gain ($E_B > 0$) by virtue of his operation. Because the average value of energy around Bob is zero before the operation, he actually extracts positive energy E_B out of the subsystem in a local ground state as *nothing*. Thus, E_B is called the teleported energy in the QET protocol and is evaluated as

$$E_B = \text{Tr}[H\rho_1] - \text{Tr}[H\rho_2]. \quad (11)$$

To derive a general formula for E_B for models of nearest neighbor interaction, let us introduce a semilocal Hermitian operator H_B as

$$H_B = T_{n_B-1} + T_{n_B} + T_{n_B+1}.$$

This is the total sum of energy density operators on which a local operation at site $n = n_B$ may have a non-trivial influence. H_B can be physically interpreted as a localized energy operator around Bob's site, satisfying $\langle g|H_B|g\rangle = 0$. Due to the locality of Alice's measurement, it is straightforwardly verified by successive use of Eqs. (7), (3) and (4) that

$$\text{Tr}[H_B\rho_1] = 0. \quad (12)$$

After the operation $U_B(\mu)$, it has been proven for the QET protocols [2] that the average value of H_B takes a negative value:

$$\text{Tr}[H_B \rho_2] < 0. \quad (13)$$

Eq. (13) describes that Bob's local operation, which enables him to extract positive energy from a subsystem with zero energy, simultaneously generates a region with negative energy density around the subsystem due to local energy conservation. It is worth to recall that the total energy of the system remains non-negative even after the emergence of the negative-energy region. Therefore the amount of energy extracted by Bob does not become larger than that injected by Alice: $E_B \leq E_A$. Because $U_B(\mu)$ is a local unitary operation at site $n = n_B$, this operation does not affect energy density of site \bar{n} with $|\bar{n} - n_B| \geq 2$, i.e.

$$\text{Tr}[T_{\bar{n}} \rho_1] - \text{Tr}[T_{\bar{n}} \rho_2] = 0 \quad (14)$$

holds for such outside sites by virtue of $[T_{\bar{n}}, U_B(\mu)] = 0$ and $U_B(\mu)^\dagger U_B(\mu) = I$. By substituting Eqs. (12) and (14) into Eq. (11), the following energy-conservation relation is directly verified:

$$E_B = -\text{Tr}[H_B \rho_2]. \quad (15)$$

This indicates that the sum of the energy gain of Bob and the negative localized energy at site n_B of the system after the operation is equal to zero, that is, the initial value of the localized energy as it should be. After simple manipulation by successively substituting Eqs. (10), (9) and (5) into Eq. (15), it can be proven that E_B takes the general form

$$E_B = -\sum_{\mu} \langle g | \Pi_A(\mu) H_B(\mu) | g \rangle, \quad (16)$$

where $H_B(\mu)$ is defined by

$$H_B(\mu) = U_B(\mu)^\dagger H_B U_B(\mu)$$

and $[\Pi_A(\mu), H_B(\mu)] = 0$ holds due to Eq. (3). Eq. (16) expresses that teleported energy E_B is equal to a sum of ground-state correlation functions of the local POVM operator $\Pi_A(\mu)$ at site $n = n_A$ and semilocal operators $H_B(\mu)$ at site $n = n_B$. If we have no ground-state entanglement, it is easy by

using Eq. (16) to check that E_B cannot be positive for any $\theta(\mu)$ and $G_B(\mu)$, as follows. For a non-entangled ground state (separable ground state) that takes the form

$$|g\rangle = \prod_n |g_n\rangle, \quad (17)$$

by using a local pure state $|g_n\rangle$ at site n , the two-point correlation function is reduced to the following product form:

$$\langle g | \Pi_A(\mu) H_B(\mu) | g \rangle = \langle g | \Pi_A(\mu) | g \rangle \langle g | H_B(\mu) | g \rangle.$$

Using $\langle g | U_B(\mu)^\dagger T_{\bar{n}} U_B(\mu) | g \rangle = 0$ with $|\bar{n} - n_B| \geq 2$, the following relation is proven:

$$\langle g | H_B(\mu) | g \rangle = \langle g | U_B(\mu)^\dagger \left(\sum_{n=n_B-1}^{n_B+1} T_n \right) U_B(\mu) | g \rangle = \langle g | U_B(\mu)^\dagger H U_B(\mu) | g \rangle. \quad (18)$$

Thus $\langle g | H_B(\mu) | g \rangle$ takes a non-negative value due to the non-negativity of H . Taking account of Eq. (6), this result means that E_B has a non-positive value for the non-entangled ground state:

$$E_B = - \sum_{\mu} p_A(\mu) \langle g | U_B(\mu)^\dagger H U_B(\mu) | g \rangle \leq 0. \quad (19)$$

However, the situation drastically changes for entangled ground states and E_B can actually take a positive value. In order to grasp the reason why positive E_B is allowed, let us consider, for instance, $U_B(\mu)$ with an infinitesimal value of $\theta(\mu)$. In this case, Eq. (16) can be expanded as

$$E_B = \sum_{\mu} \theta(\mu) \langle g | \Pi_A(\mu) \dot{G}_B(\mu) | g \rangle + O(\theta^2), \quad (20)$$

where $\dot{G}_B(\mu)$ is a semilocal Hermitian operator around site $n = n_B$ defined by

$$\dot{G}_B(\mu) = i [H_B, G_B(\mu)].$$

Since $[\Pi_A(\mu), \dot{G}_B(\mu)] = 0$ is guaranteed by Eq. (3), the correlation function $\langle g | \Pi_A(\mu) \dot{G}_B(\mu) | g \rangle$ takes a real number. For an entangled ground state $|g\rangle$

satisfying $\langle g | \Pi_A(\mu) \dot{G}_B(\mu) | g \rangle \neq 0$ for some μ , Eq. (20) reveals that E_B is capable of taking a positive value:

$$E_B > 0$$

by appropriately choosing the sign of $\theta(\mu)$ so as to make $\theta(\mu) \langle g | \Pi_A(\mu) \dot{G}_B(\mu) | g \rangle$ positive. It should be stressed that rather general spin-chain models with entangled ground states are able to satisfy the condition of non-vanishing two-point correlation. Hence a very wide class of spin-chain models are available for QET with E_B positive. In conventional QET protocols [2], the sign and magnitude of $\theta(\mu)$ are usually determined in order to maximize the positive value of E_B .

Eq. (16) directly connects E_B with two-point correlation functions. By definition, the correlation functions provide information about how the zero-point fluctuation at site $n = n_A$ is correlated with that at site $n = n_B$. This correlation is caused unquestionably by the ground-state entanglement. Thus, it is natural to expect that E_B has a nontrivial relation to ground-state entanglement in general, and this is found to be true. The next section discusses how the ground-state entanglement entropy between site $n = n_A$ and its complementary region is lower bounded by a positive quadratic function of teleported energy of a general QET protocol.

Before closing this section, a comment is given on the ground-state entanglement. Our understanding of many-body quantum entanglement is not enough yet. We have a lot of entanglement measures, which advantages are indeed verified in some cases [12]. In the bipartite entanglement case with an energy-sender subsystem A and an energy-receiver subsystem B of QET, several entanglement measures including the negativity and the log-negativity can be explicitly computed from a reduced density operator ρ_{AB} of the subsystems. It is known that, even for many-body systems at criticality at zero temperature, such a bipartite entanglement measure is calculated as a product of a power law and an exponential decay in terms of the separation between A and B [13] [11]. Thus the bipartite entanglement, that would be a resource of QET, becomes negligibly small for a large separation beyond a typical length of the system. However, the amount of teleported energy from A to B just obeys a power-law decay as the distance becomes large [2] [7]. Thus the long-distance QET remains effective even though the bipartite entanglement is not available. This superficial paradox is resolved by noting that the mutual information between A and B decays in a power law

manner in contrast to the bipartite entanglement. QET can be performed only by use of this mutual information shared by A and B . The bipartite entanglement of the two subsystems is not necessary. However, it should be stressed that this correlation of A and B described by the mutual information is actually generated by not only the bipartite entanglement but also multipartite entanglements in the ground state [14]. If the ground state is an exactly separable (non-entangled) state which takes a product form of pure states of all subsystems, we do not have such a correlation between them at all. Thus it is a quite natural attempt to introduce a notion of ‘the mother entanglement’, which gives birth to the mutual information between A and B for QET. Then what are the most appropriate entanglement measure for the description of this mother entanglement? Unfortunately, this remains a serious open problem. However it can be said, at least, that the entanglement entropy $S_{ent}(A, \bar{A})$ of A and its complement \bar{A} , which includes B , precisely captures the mother entanglement property. In fact, if $S_{ent}(A, \bar{A}) = 0$, no mutual information of A and B is generated and QET does not work at all. In this sense, this entanglement entropy is truly a resource of QET for the ground-state case. Therefore, in the next section, we adopt $S_{ent}(A, \bar{A})$ in order to describe how much entanglement the spin-chain systems possess as a QET resource at zero temperature. At the end a remark is appended for finite temperature cases. It has been shown very recently that not quantum entanglement but quantum discord [15] becomes a resource for protocols of finite-temperature QET [16]. For the Ising spin model composed of two qubits in the presence of transverse magnetic field, we have a critical temperature above which entanglement between the qubits in the thermal state completely vanishes, though the quantum discord (thermal discord) remains nonzero. By utilizing information shared via the quantum discord, a high-temperature QET protocol for the two qubits can extract more energy out of one qubit in the thermal state than that extracted only by use of local operations.

3 Ground-State Entanglement Bound in Terms of Teleported Energy

Let us consider a ground state $|g\rangle$ of a general spin-chain model. Let region A be composed of a single site $n = n_A$ and region B be composed of three sites with $n = n_B - 1, n_B, n_B + 1$ satisfying Eq. (3). The reduced state for A is given by $\rho_A = \text{Tr}_{\bar{A}}[|g\rangle\langle g|]$, that for B by $\rho_B = \text{Tr}_{\bar{B}}[|g\rangle\langle g|]$, and that for $A \cup B$ by $\rho_{AB} = \text{Tr}_{\overline{A \cup B}}[|g\rangle\langle g|]$, where the bar for a set means complement of the set. Herein, we analytically show, for a general spin-chain model, that entanglement entropy between A and \bar{A} of a ground state is lower bounded by a positive quadratic function of energy E_B teleported from A to B in an arbitrary QET protocol. To derive the inequality, we first focus on not entanglement entropy itself but instead mutual information $I_{A:B}$ between A and B defined as

$$I_{A:B} = S_A + S_B - S_{AB}, \quad (21)$$

where $S_A = S(\rho_A)$, $S_B = S(\rho_B)$, $S_{AB} = S(\rho_{AB})$, and $S(\rho)$ is the von Neumann entropy of ρ :

$$S(\rho) = -\text{Tr}[\rho \ln \rho].$$

When $|g\rangle$ is an entangled state, the mutual information $I_{A:B}$ may take a positive value. It is first noted that the following inequality is proven: For $I_{A:B}$ in Eq. (21) and E_B in Eq. (16),

$$I_{A:B} \geq \frac{|E_B + \langle H \rangle|^2}{2 \|H_B\|^2}, \quad (22)$$

where $\langle H \rangle$ is defined by

$$\langle H \rangle = \sum_{\mu} p_A(\mu) \langle g | U_B(\mu)^\dagger H U_B(\mu) | g \rangle. \quad (23)$$

$\langle H \rangle$ can be interpreted as excitation energy of the system after performing a probabilistic operation $U_B(\mu)$ with its probability $p_A(\mu)$ to the ground state. Unless $U_B(\mu) = I$ for each μ , $\langle H \rangle$ must take a positive value owing to the passivity of the ground state. $\|H_B\|$ in Eq. (22) stands for the matrix norm of H_B given by the maximum absolute value of the eigenvalue of H_B :

$$\|H_B\| = \max \{ |\varepsilon_B| : H_B |\varepsilon_B\rangle = \varepsilon_B |\varepsilon_B\rangle \}.$$

The proof of Eq. (22) is as follows: Let us think a pointer system A' of Alice's measurement device. Consider a complete orthogonal vector basis

$\{|\mu_{A'}\rangle : \langle\mu_{A'}|\mu'_{A'}\rangle = \delta_{\mu\mu'}\}$ in a Hilbert space of A' corresponding to the measurement output $\{\mu\}$ of measurement operator $M_A(\mu)$. Before the measurement, assume that the pointer state is in a pure state $|0_{A'}\rangle$. The total initial state of the composite system of A' , A , and B before the measurement of Alice is given by

$$\Phi_{A'AB} = |0_{A'}\rangle\langle 0_{A'}| \otimes \rho_{AB}.$$

Because $|0_{A'}\rangle$ is a pure state and A' has no correlation with A and B , mutual information $I_{A'A:B}$ between $A' \cup A$ and B of $\Phi_{A'AB}$ is equal to $I_{A:B}$ of $|g\rangle\langle g|$ between A and B :

$$I_{A'A:B} = I_{A:B}.$$

Let us consider a quantum operation Γ for $\Phi_{A'AB}$ that describes the measurement of Alice and satisfies

$$\Gamma[\Phi_{A'AB}] = \sum_{\mu} |\mu_{A'}\rangle\langle\mu_{A'}| \otimes M_A(\mu)\rho_{AB}M_A(\mu)^\dagger.$$

After performing the operation, we discard subsystem A and define a reduced state $\rho_{A'B}$ defined by

$$\rho_{A'B} = \text{Tr}_A[\Gamma[\Phi_{A'AB}]].$$

Note that $\rho_B = \text{Tr}_{A'}[\rho_{A'B}]$ holds because of the locality of Alice's measurement. Taking account of this relation, the mutual information $I_{A':B}$ between A' and B after the manipulation is computed as

$$I_{A':B} = S(\rho_{A'}) + S(\rho_B) - S(\rho_{A'B}),$$

where

$$\rho_{A'} = \sum_{\mu} p_A(\mu) |\mu_{A'}\rangle\langle\mu_{A'}|, \quad (24)$$

$$\rho_{A'B} = \sum_{\mu} |\mu_{A'}\rangle\langle\mu_{A'}| \otimes \text{Tr}_A[\Pi_A(\mu)\rho_{AB}]. \quad (25)$$

Let us define a μ -dependent post-measurement state $\rho_B(\mu)$ as

$$\rho_B(\mu) = \frac{1}{p_A(\mu)} \text{Tr}_A[\Pi_A(\mu)\rho_{AB}] = \frac{1}{p_A(\mu)} \text{Tr}_B[\Pi_A(\mu)|g\rangle\langle g|]. \quad (26)$$

Then, we are able to rewrite Eq. (25) in a transparent form that describes the perfect correlation between A' and B of the post-measurement state as

$$\rho_{A'B} = \sum_{\mu} p_A(\mu) |\mu_{A'}\rangle\langle\mu_{A'}| \otimes \rho_B(\mu). \quad (27)$$

It is a well-known monotonicity property that both quantum operation and discard of subsystems never increase mutual information [17]. This monotonicity can be proven by use of strong subadditivity of the von Neumann entropy [18]. Therefore, the following relation holds:

$$I_{A'A:B} \geq I_{A':B}.$$

Because $I_{A:B} = I_{A'A:B}$, this implies the following inequality:

$$I_{A:B} \geq I_{A':B}. \quad (28)$$

Here, it is worthwhile to note a useful relation of relative entropy [19] that

$$S(\rho||\varphi) \geq \frac{1}{2} (\|\rho - \varphi\|_1)^2, \quad (29)$$

where $S(\rho||\varphi) = \text{Tr}[\rho \ln \rho] - \text{Tr}[\rho \ln \varphi]$ for two quantum states ρ and φ , and $\|\rho - \varphi\|_1$ is the trace norm of $\rho - \varphi$ given by $\|\rho - \varphi\|_1 = \text{Tr} \left[\sqrt{(\rho - \varphi)^2} \right]$. The proof of Eq. (29) is outlined in Appendix 1. Since $I_{A':B}$ is expressed by using relative entropy as

$$I_{A':B} = S(\rho_{A'B}||\rho_{A'}\rho_B),$$

the following inequality is satisfied:

$$I_{A':B} = S(\rho_{A'B}||\rho_{A'}\rho_B) \geq \frac{1}{2} (\|\rho_{A'B} - \rho_{A'}\rho_B\|_1)^2.$$

Because

$$\|X\|_1 \geq \frac{|\text{Tr}[XY]|}{\|Y\|} \quad (30)$$

holds for arbitrary Hermitian operators X and Y as proven in Appendix 2,

$$\frac{1}{2} (\|\rho_{A'B} - \rho_{A'}\rho_B\|_1)^2 \geq \frac{|\text{Tr}[\rho_{A'B}M_{A'B}] - \text{Tr}[\rho_{A'}\rho_B M_{A'B}]|^2}{2 \|M_{A'B}\|^2}$$

holds for an arbitrary Hermitian operator $M_{A'B}$ of the composite system of A' and B . This inequality is provided by Wolf et al [20] in local operator product cases: $M_{A'B} = M_{A'} \otimes M_B$. In later discussion, we fix $M_{A'B}$ in a

specific form by use of Bob's operation $U_B(\mu) = \exp(-i\theta(\mu)G_B(\mu))$. Let us introduce a non-local unitary operator

$$U_{A'B} = \exp\left(-i \sum_{\mu} \theta(\mu) |\mu_{A'}\rangle \langle \mu_{A'}| G_B(\mu)\right),$$

This operators satisfy

$$U_{A'B} |\mu_{A'}\rangle = U_B(\mu) |\mu_{A'}\rangle. \quad (31)$$

By using $U_{A'B}$, the operator $M_{A'B}$ is defined as follows:

$$M_{A'B} = U_{A'B}^\dagger H_B U_{A'B}. \quad (32)$$

Because $\|M_{A'B}\| = \|H_B\|$, we are able to derive the following inequality:

$$I_{A':B} \geq \frac{\left| \text{Tr} \left[\rho_{A'B} U_{A'B}^\dagger H_B U_{A'B} \right] - \text{Tr} \left[\rho_{A'} \rho_B U_{A'B}^\dagger H_B U_{A'B} \right] \right|^2}{2 \|H_B\|^2}.$$

Using Eqs. (27), (31), (26), (16) successively, it can be shown that $\text{Tr} \left[\rho_{A'B} U_{A'B}^\dagger H_B U_{A'B} \right]$ is equal to $-E_B$ as follows:

$$\begin{aligned} & \text{Tr} \left[\rho_{A'B} U_{A'B}^\dagger H_B U_{A'B} \right] \\ &= \sum_{\mu} p_A(\mu) \text{Tr}_B \left[\rho_B(\mu) \langle \mu_{A'} | U_{A'B}^\dagger H_B U_{A'B} | \mu_{A'} \rangle \right] \\ &= \sum_{\mu} p_A(\mu) \text{Tr}_B \left[\rho_B(\mu) U_B(\mu)^\dagger H_B U_B(\mu) \right] \\ &= \sum_{\mu} \text{Tr}_B \left[\text{Tr}_{\bar{B}} [\Pi_A(\mu) |g\rangle \langle g|] U_B(\mu)^\dagger H_B U_B(\mu) \right] \\ &= -E_B. \end{aligned}$$

Similarly, using Eqs. (24), (31), (18) and (23), it can be proven that $\text{Tr} \left[\rho_{A'} \rho_B U_{A'B}^\dagger H_B U_{A'B} \right]$

is equal to $\langle H \rangle$ as follows:

$$\begin{aligned}
& \text{Tr} \left[\rho_{A'} \rho_B U_{A'B}^\dagger H_B U_{A'B} \right] \\
&= \sum_{\mu} p_A(\mu) \langle g | U_B(\mu)^\dagger H_B U_B(\mu) | g \rangle \\
&= \sum_{\mu} p_A(\mu) \langle g | U_B(\mu)^\dagger H U_B(\mu) | g \rangle \\
&= \langle H \rangle.
\end{aligned}$$

Therefore, we obtain the following inequality:

$$I_{A':B} \geq \frac{|E_B + \langle H \rangle|^2}{2 \|H_B\|^2},$$

and Eq. (22) is proven because of Eq. (28). The result simultaneously yields another inequality:

$$I_{A:B} \geq \frac{E_B^2}{2 \|H_B\|^2}.$$

This implies that performing the QET with teleported energy E_B requires the mutual information more than $E_B^2/(2 \|H_B\|^2)$. Thus it can be said that the mutual information $I_{A:B}$ is a resource of QET. However, as emphasized in the last paragraph of section 2, $I_{A:B}$ is generated by the ground-state entanglement. Therefore it is quite natural to rewrite the result in terms of the entanglement entropy $S_{ent}(A, \bar{A})$. Since $\bar{A} \supset B$ and the monotonicity of mutual information holds in discarding subsystems of no interest [17],

$$I_{A:\bar{A}} = S(\rho_A) + S(\rho_{\bar{A}}) - S(|g\rangle\langle g|) \geq I_{A:B}, \quad (33)$$

where $\rho_{\bar{A}} = \text{Tr}_A[|g\rangle\langle g|]$, is also satisfied. Owing to purity of the ground state, $S(|g\rangle\langle g|) = 0$ and $S(\rho_A) = S(\rho_{\bar{A}})$. Thus, Eq. (33) yields the following inequality:

$$S_{ent}(A, \bar{A}) \geq \frac{1}{2} I_{A:B}, \quad (34)$$

where $S_{ent}(A, \bar{A})$ is given by

$$S_{ent}(A, \bar{A}) = S(\rho_A) = \frac{1}{2} I_{A:\bar{A}}.$$

From Eqs. (34) and (22), we finally obtain one of our main results:

$$S_{ent}(A, \bar{A}) \geq \frac{|E_B + \langle H \rangle|^2}{4 \|H_B\|^2}. \quad (35)$$

The equality of Eq. (35) is attained for spin-chain models with separable ground states with its form in Eq. (17) because $S_{ent}(A, \bar{A}) = 0$ and $E_B + \langle H \rangle = 0$ as seen in Eq. (19). Eq. (35) gives a lower bound for the ground-state entanglement entropy $S_{ent}(A, \bar{A})$ for general spin-chain QET protocols with $E_B > 0$. Trivially, the following inequality holds for arbitrary QET protocols since both E_B and $\langle H \rangle$ in Eq. (35) are positive:

$$S_{ent}(A, \bar{A}) \geq \frac{E_B^2}{4 \|H_B\|^2}. \quad (36)$$

If we know the value of E_B in a specific QET protocol, Eq. (36) provides a lower bound of ground-state entanglement entropy from an operational viewpoint. The result of Eq. (36) strongly supports the conjecture that a large amount of ground-state entanglement $S_{ent}(A, \bar{A})$ is required as an evident physical resource to perform a QET protocol when the amount of teleported energy E_B is large.

It is a rather straightforward extension to consider larger regions for energy sender A and receiver B . Let A be Alice's region with $n = n_A - l_A \sim n_A + l_A$ and B be Bob's region with $n = n_B - l_B \sim n_B + l_B$ by setting l_A and l_B to be positive integers and by assuming that $|n_A - n_B| \geq 2 + l_A + l_B$. Now $M_A(\mu)$ is a measurement operator acting on a composite Hilbert subspace of spins in A , and $U_B(\mu)$ is a unitary operator acting on a composite Hilbert subspace of spins with $n = n_B - l_B + 1 \sim n_B + l_B - 1$. Then, H_B is redefined as

$$H_B = \sum_{n=n_B-l_B}^{n_B+l_B} T_n.$$

Even after such an extension, our results in Eqs. (16), (23), (35), and (36) still hold. This extension may deepen our understanding of QET and ground-state entanglement itself. For example, it turns out that Eq. (36) provides a universal upper bound of the ratio $E_B / \|H_B\|$ independent of the number $2l_B - 1$ of energy extraction points of B and the detail of QET protocols. For an arbitrary fixed subsystem A , E_B may be expected to become large as l_B becomes large, but $\|H_B\|$ becomes larger as well and the ratio $E_B / \|H_B\|$ never exceeds $2S_{ent}(A, \bar{A})^{1/2}$. Note that $E_B / \|H_B\| \leq 1$ holds by definition

of the matrix norm. Thus the upper bound $2S_{ent}(A, \bar{A})^{1/2}$ provides valuable information about the teleported energy when $S_{ent}(A, \bar{A}) < 1/4$. The extension also allows us to treat A and \bar{A} symmetrically even if the sizes are different. We are able to exchange the roles of A and \bar{A} in the QET protocols: Now \bar{A} is the measured system, and a subsystem \tilde{A} of A belonging to $[n_A - l_A + 1, n_A + l_A - 1]$ is the controlled system dependent on the measurement result, out of which teleported energy $E_{\tilde{A}}$ is extracted. Under such an exchange, the entanglement entropy remains unchanged because of the ground-state purity:

$$S_{ent}(\bar{A}, A) = S_{ent}(A, \bar{A}).$$

The left hand side of Eq. (36) is also unchanged. Therefore the following inequality holds:

$$S_{ent}(A, \bar{A}) \geq \frac{E_{\tilde{A}}^2}{4 \|H_{\tilde{A}}\|^2},$$

where $H_{\tilde{A}}$ denotes the localized energy operator of \tilde{A} . This provides another lower bound of the ground-state entanglement entropy.

4 Summary and Discussion

We considered the ground state of a general model of a spin chain with nearest neighbor interaction to analyze an arbitrary QET protocol. A universal inequality in Eq. (35) is proven. This inequality implies that ground-state entanglement $S_{ent}(A, \bar{A})$ between the energy sender's region A and its complementary region \bar{A} , which includes the energy receiver's region B , is lower bounded by a positive quadratic function of teleported energy E_B . The obtained results still hold even in the extended settings for large A and B . The result of Eq. (36), derived from Eq. (35), strongly supports the general conjecture that a large amount of ground-state entanglement is required as an evident physical resource to perform a QET protocol when the amount of teleported energy is large.

The results in this paper are expected to deepen our understanding of the energy density fluctuation in condensed matter systems from a new perspective of quantum information theory. The teleported energy originally emerges from the zero-point fluctuation in the ground state. Hence the amount of the energy and its distance dependence may lead us to fundamental relations of the condensed matter systems, which can be naturally termed ‘the fluctuation-information relation’. For instance, if a state tomography of the ground state is experimentally achieved, we are able to evaluate the entanglement entropy $S_{ent}(A, \bar{A})$. Then the inequality of Eq. (36) yields an upper bound of E_B that is closely related to the amplitude of energy density fluctuation in the ground state.

In this paper, we consider one-dimensional spin chain models. The extension to higher-dimensional lattice models with nearest neighbor interaction can be achieved easily and the same results are obtained, especially the result of Eq. (36) holds. As stressed in [20], the area law of entanglement entropy $S_{ent}(A, \bar{A})$ of a compact region A and its complement \bar{A} in terms of the boundary area is generally proven for the ground states of the models, and interestingly alludes to relations between the entanglement entropy and the holographic principle of black hole physics. When we perform a measurement of A and, by a QET protocol, extract the corresponding teleported energy E_B from an outside region B that almost overlaps \bar{A} except a buffer area shared with A , an upper bound of the teleported energy

$$E_B \leq 2 \|H_B\| S_{ent}(A, \bar{A})^{1/2}$$

is derived from Eq. (36). This suggests a nontrivial area dependence of E_B . The analyses of the dependence are of much interest for black hole physics and may provide a more profound insight into the holographic principle. The results will be reported elsewhere.

Acknowledgments

I would like to thank Holger F. Hofmann for giving me valuable comments about the previous manuscript. This research has been partially supported by the Global COE Program of MEXT, Japan, and the Ministry of Education, Science, Sports and Culture, Japan, under Grant No. 21244007.

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Appendix 1

In this appendix, a proof outline of Eq. (29) in [19] is shown. As preparation, let us first consider a function $f_x(y)$ of x and y with $0 \leq x \leq y \leq 1$ defined by

$$f_x(y) = x \ln \left(\frac{x}{y} \right) + (1-x) \ln \left(\frac{1-x}{1-y} \right) - 2(y-x)^2.$$

The partial derivative in terms of y is found to be non-negative:

$$\partial_y f_x(y) = \frac{(1-2y)^2}{y(1-y)} (y-x) \geq 0.$$

Therefore, the minimum value of $f_x(y)$ in terms of y for a fixed value of x is zero:

$$\min_{x \leq y \leq 1} f_x(y) = f_x(x) = 0.$$

Hence, $f_x(y)$ is non-negative and the inequality

$$2(x-y)^2 \leq x \ln \left(\frac{x}{y} \right) + (1-x) \ln \left(\frac{1-x}{1-y} \right) \quad (37)$$

holds for $0 \leq x \leq y \leq 1$. Next, let us consider the spectral decomposition of $\rho - \varphi$, where ρ and φ are density operators of quantum states:

$$\rho - \varphi = \sum_{\lambda} \lambda P(\lambda).$$

Here, λ is an eigenvalue of $\rho - \varphi$ and $P(\lambda)$ is its corresponding projective operator. Let us define a projective operator P_+ by a sum of $P(\lambda)$ with non-negative eigenvalues:

$$P_+ = \sum_{\lambda \geq 0} P(\lambda).$$

Further, let us introduce a projection operator P_- as the complement of P_+ :

$$P_- = \sum_{\lambda < 0} P(\lambda) = I - P_+.$$

We define emergent probabilities of the ideal measurement result of P_{\pm} for the two states ρ and φ as follows:

$$\begin{aligned} p_{\pm} &= \text{Tr} [\rho P_{\pm}] , \\ q_{\pm} &= \text{Tr} [\varphi P_{\pm}] . \end{aligned}$$

Then, the trace norm $\|\rho - \varphi\|_1$ is computed as

$$\begin{aligned} \|\rho - \varphi\|_1 &= \sum_{\lambda} |\lambda| = \sum_{\lambda \geq 0} \lambda - \sum_{\lambda < 0} \lambda \\ &= \text{Tr} [(\rho - \varphi) P_+] - \text{Tr} [(\rho - \varphi) P_-] \\ &= 2 \text{Tr} [(\rho - \varphi) P_+] \\ &= 2 (p_+ - q_+) . \end{aligned}$$

Because the inequality

$$2 (p_+ - q_+)^2 \leq p_+ \ln \left(\frac{p_+}{q_+} \right) + p_- \ln \left(\frac{p_-}{q_-} \right)$$

is generally satisfied according to Eq. (37), the following relation holds:

$$\frac{1}{2} (\|\rho - \varphi\|_1)^2 = 2 (p_+ - q_+)^2 \leq p_+ \ln \left(\frac{p_+}{q_+} \right) + p_- \ln \left(\frac{p_-}{q_-} \right) = S_c(p||q),$$

where $S_c(p||q)$ is the classical relative entropy of p and q . Because of the monotonicity of the relative entropy [19], the classical relative entropy $S_c(p||q)$ is upper bounded by the quantum relative entropy $S(\rho||\varphi)$:

$$S_c(p||q) \leq S(\rho||\varphi).$$

Therefore, we obtain Eq. (29).

Appendix 2

In this appendix, we prove a standard triangular inequality of the trace norm and the matrix norm in Eq. (30). Let X and Y be Hermitian operators. Let us introduce a spectral decomposition of X and Y as

$$X = \sum_n x_n |x_n\rangle \langle x_n|, \quad (38)$$

$$Y = \sum_m y_m |y_m\rangle \langle y_m|. \quad (39)$$

Then, the matrix norm of Y is written as

$$\|Y\| = \max_m |y_m| = |y|_{\max}.$$

Eqs. (38) and (39) yield the following relation:

$$\begin{aligned} \frac{|\text{Tr}[XY]|}{\|Y\|} &= \left| \sum_n x_n \sum_m \frac{y_m}{|y|_{\max}} |\langle x_n | y_m \rangle|^2 \right| \\ &\leq \sum_n |x_n| \left| \sum_m \frac{y_m}{|y|_{\max}} |\langle x_n | y_m \rangle|^2 \right|. \end{aligned} \quad (40)$$

Because

$$0 \leq \frac{|y_m|}{|y|_{\max}} \leq 1$$

holds, the right-hand-side term in Eq. (40) is upper bounded by $\|X\|_1$ as follows:

$$\begin{aligned} &\sum_n |x_n| \left| \sum_m \frac{y_m}{|y|_{\max}} |\langle x_n | y_m \rangle|^2 \right| \\ &\leq \sum_n |x_n| \left| \sum_m |\langle x_n | y_m \rangle|^2 \right| \\ &= \sum_n |x_n| = \|X\|_1. \end{aligned}$$

Thus, we obtain Eq. (30).